INPUT IMPEDANCE OF A SPHERICAL FERRITE ANTENNA WITH A LATITUDINAL CURRENT

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TABLE OF CONTENTS

ABSTRACT

An expression for the input impedance of a permeable spheroid with a latitudinal surface current is derived and evaluated for the case of a small sphere. Formulas for the quality factor and radiation efficiency are presented, assuming simple current distributions. The extrapolation of the results for the sphere for application to spheroids is considered.

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge the suggestions and comments of V. H. Rumsey and R. H. DuHamel, who also originally suggested the sphere problem.

GLOSSARY OF SYMBOLS

A	auxiliary wave function ($A = h_\phi E_\phi$)
D	static demagnetization factor
E	electric field intensity
$\mathcal{E}$	radiation efficiency
I_i	input current
$I(n)$	$= \int dv \, \frac{h_v}{h_u h_\phi} V_n$
J	current density
$J_{n+\frac{1}{2}}$	Bessel function of first kind
$\hat{J}_n$	spherical Bessel function (See Eq. 46)
$\hat{J}_n'$	derivative of spherical Bessel function
K	surface current density
$K(0)$	K at equator (i.e., at $z = 0$)
$\hat{K}_n$	spherical Hankel function (See Eq. 47)
$\hat{K}_n'$	derivative of $\hat{K}_n$
$N_{n+\frac{1}{2}}$	Bessel function of second kind
P_n	Legendre polynomial
P_n^1	associated Legendre polynomial
Q	equality factor $\frac{\text{Im } Z}{\text{Re } Z}$
R_c	effective series resistance to represent ferrite losses
R_R	radiation resistance
U_n	a solution of Eq. 11
V_n	a solution of Eq. 12
Z	input impedance

a	radius of sphere
dS	element of surface
ds	element of arc
$f(h_u, h_v, h_\phi)$	$= [h_u, h_v, h_\phi]^{-1/2}$
$g(v)$	$= \left[\frac{h_v}{h_u h_\phi}\right]^{1/2}$
$g_1(v)$	$= \frac{h_\phi h_v}{h_u}$
$g_2(v)$	$= \frac{h_\phi h_u}{h_v}$
h_u, h_v, h_ϕ	metrical coefficients for the orthogonal coordinates u, v, and ϕ, respectively
j	$\sqrt{-1}$
A_{V_n}	normalization factor for the functions V_n
α_n	coefficients in the series expansion of the current
β	$= \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$
ε	dielectric constant
K_m	relative permeability (may be complex)
K_m'	real component of K_m
K_m''	imaginary component of K_m
λ	wavelength of radiation
μ	permeability
ω	circular frequency

1. INTRODUCTION

We will consider the problem of a non-conducting (but not lossless) permeable sphere which is enclosed by a distribution of surface currents. The currents are assumed:

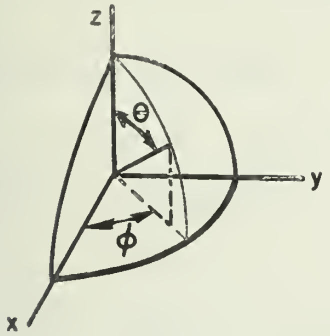

Figure 1

a) to have no ϕ variation

b) to have only a ϕ direction, and

c) to be true surface currents, $\underline{K}$, (in the sense that

$$\lim_{\substack{\underline{J}\to\infty \\ dr\to 0}} \underline{J}dr = \underline{K},$$

where $\underline{J}$ is the volume current density)

confined to the surface of a sphere, $r = a$. It is also assumed that the generator of these currents does not disturb the symmetry. Under these conditions, the input impedance can be found from the expression

$$|I_i|^2 Z = -\iint E_\phi K^* \, dS$$

in which I_i is the input current, Z is the impedance, E_ϕ is the ϕ component of the electric field which is generated by the currents in the presence of the ferrite and K is the surface current density. The main problem is, therefore, to determine E_ϕ.

The surface current distribution is an arbitrary function of θ. For convenience, the arbitrary function will be represented by a series of associated Legendre polynomials.

As a matter of fact, since it involves little extra difficulty, we will not at first restrict the discussion to spheres. Instead we will present a more general development which will apply to spheroids. Thus,

we will discuss a problem in which the surface of the ferrite coincides with some surface, u = constant, of a system of orthogonal coordinates (u, v, ϕ) in which the surfaces u = constant and v = constant are generated by rotations about the z-axis. The specific development for the case of a sphere is presented in Appendix A as a detailed illustration.

The problem is admittedly an academic one. While the answer which is obtained is correct if a current is established as specified, no consideration is given here to the question of how such a current distribution would be established. The results are nevertheless useful as approximations to solutions for small antennas since in this case the current should have negligible ϕ-variation. Furthermore, some flexibility is retained in the formulation in that any θ (or v) variation can be represented. Thus it is possible to consider the problem of a small ferrite sphere wrapped with a conducting ribbon at its equator, or enclosed by the turns of a single layer winding of fine wire on its surface. In the latter case, the surface density of current is interpreted as the number of turns per unit length of arc (meridian) times the current in each turn.

2. FIELD EQUATIONS

The electromagnetic fields must satisfy Maxwell's equations and the boundary conditions. A form of the field equations applicable to this problem[1] (assuming $e^{j\omega t}$ time convention) is

$$\frac{\partial(h_\phi E_\phi)}{\partial v} = -j\omega\mu\ h_v h_\phi H_u \qquad (1) \qquad \frac{\partial(h_\phi H_\phi)}{\partial v} = j\omega\varepsilon\ h_v h_\phi E_u$$

$$\frac{\partial(h_\phi E_\phi)}{\partial u} = j\omega\mu\ h_\phi h_u H_v \qquad (2) \qquad \frac{\partial(h_\phi H_\phi)}{\partial u} = -j\omega\varepsilon\ h_\phi h_u E_v$$

$$\frac{\partial(h_v H_v)}{\partial u} - \frac{\partial(h_u H_u)}{\partial v} = j\omega\varepsilon\ h_u h_v E_\phi \qquad (3) \qquad \frac{\partial(h_v E_v)}{\partial u} - \frac{\partial(h_u E_u)}{\partial v} = -j\omega\mu\ h_u h_v H_\phi$$

wherein use has been made of the assumed symmetry to remove the derivatives with respect to ϕ; h_u, h_v, h_ϕ are the metrical coefficients for the orthogonal coordinate system. Note that the equations consist of two independent sets, one involving only H_u, H_v, and E_ϕ, and the other involving only E_u, E_v, and H_ϕ. We need concern ourselves only with the first set (the left column above) since we require only E_ϕ for the impedance calculation. (As a matter of fact, it will be apparent from the solution of the set on the left that the solution of the other set which fits the boundary conditions gives $E_u = E_v = 0 = H_\phi$.)

1. Schelkunoff, S. A. *Advanced Antenna Theory*, John Wiley & Sons, 1952. p. 4.

3. SOLUTION OF THE FIELD EQUATIONS

An auxiliary wave function helps to solve the equations. We introduce A, so defined that $A = h_\phi E_\phi$. Then,

$$H_u = -\frac{1}{j\omega\mu\, h_v h_\phi}\frac{\partial A}{\partial v} \tag{4}$$

$$H_v = \frac{1}{j\omega\mu\, h_\phi h_u}\frac{\partial A}{\partial u} \tag{5}$$

$$E_\phi = A/h_\phi \; . \tag{6}$$

The insertion of Eqs. 4, 5, and 6 into Eq. 3 gives

$$\frac{\partial}{\partial u}\left(\frac{h_v}{h_\phi h_u}\frac{\partial A}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u}{h_\phi h_v}\frac{\partial A}{\partial v}\right) + \omega^2\mu\varepsilon\left(\frac{h_u h_v}{h_\phi}\right)A = 0.$$

A substitution $A = U(u)\,V(v)$ results in

$$\frac{\partial}{\partial u}\left(\frac{h_v}{h_\phi h_u}\right)V\frac{\partial U}{\partial v} + \frac{h_v}{h_\phi h_u}V\frac{\partial^2 U}{\partial u^2} + \frac{\partial}{\partial v}\left(\frac{h_u}{h_v h_\phi}\right)U\frac{\partial V}{\partial v}$$
$$+\left(\frac{h_u}{h_v h_\phi}\right)U\frac{\partial^2 V}{\partial v^2} + \omega^2\mu\varepsilon\left(\frac{h_u h_v}{h_\phi}\right)UV = 0 \; . \tag{7}$$

If the variables are separable in the coordinate system selected, then the ratios of the coefficients in Eq. 7 must be such that the division of the equation by $UVf(u,v)$ makes each term independent of *either* u *or* v. A sufficient condition for the separation of the variables in Eq. 7 is then that the metrical coefficients satisfy

$$\left(\frac{h_v}{h_\phi h_u}\right) = [g_2(v)]^{-1} \text{ (or more generally } = [g_2(v)g_4(u)]^{-1}) \tag{8}$$

$$\left(\frac{h_u}{h_v h_\phi}\right) = [g_1(u)]^{-1} \text{ (or more generally } = [g_1(u)g_3(v)]^{-1}) \tag{9}$$

$$\left(\frac{h_u h_v}{h_\phi}\right) = \frac{f_1(u) + f_2(v)}{g_1(u)\; g_2(v)} \tag{10}$$

in which the f's and g's are functions of the single variable indicated. With these restrictions on the metrical coefficients, a multiplication by $g_1(u)\; g_2(v)/UV$ separates the variables and gives the two equations (employing the less general conditions)

$$g_1(u)\; U'' + (\omega^2\mu\varepsilon\; f_1(u) - k)\; U = 0 \tag{11}$$

and

$$g_2(v)\; V'' + \{k + \omega^2\mu\varepsilon\; f_2(v)\}\; V = 0\;, \tag{12}$$

where k is the separation constant. Thus, if the functions which satisfy Eqs. 11 and 12 constitute complete orthogonal sets, the general solution of the field equations (1, 2, and 3) is given by

$$H_u = -\frac{1}{j\omega\mu\; h_v h_\phi} \sum_k a_k\; U_k\; \frac{dV_k}{dv} \tag{13}$$

$$H_v = \frac{1}{j\omega\mu\; h_\phi h_u} \sum_k a_k\; \frac{dU_k}{du}\; V_k \tag{14}$$

$$E_\phi = \frac{1}{h_\phi} \sum_k a_k\; U_k\; V_k \tag{15}$$

as may be shown by first writing down Eq. 15 as the general solution for E_ϕ, and then employing Eqs. 1 and 2 to obtain explicit expressions for H_u and H_v.

4. BOUNDARY CONDITIONS AND THE DETERMINATION OF THE COEFFICIENTS

The surface u = constant = u_0 which coincides with the surface of the ferrite divides space into two regions. Let the ferrite have permeability μ_1 (may be complex) and the region outside $u = u_0$ have permeability μ_2 (assumed real). On the surface $u = u_0$, the boundary conditions on the tangential components of the fields require that

$$E_{\phi 1}(u_0) = E_{\phi 2}(u_0) \tag{16}$$

and

$$H_{v2}(u_0) - H_{v1}(u_0) = K_\phi . \tag{17}$$

Let

$$E_{\phi 1} = \frac{1}{h_\phi} \sum_n a_n U_{n1} V_n \tag{18}$$

and

$$E_{\phi 2} = \frac{1}{h_\phi} \sum_n b_n U_{n2} V_n \tag{19}$$

where U_{n1} is a solution of Eq. 11, which is finite at all points inside the ferrite, and U_{n2} is a solution of Eq. 11 which represents outgoing waves. Then

$$H_{v1} = \frac{1}{j\omega\mu_1 h_\phi h_u} \sum_n a_n \frac{d U_{n1}}{du} V_n \tag{20}$$

$$H_{v2} = \frac{1}{j\omega\mu_2 h_\phi h_u} \sum_n b_n \frac{d U_{n2}}{du} V_n . \tag{21}$$

In view of the form of the expressions for the magnetic field, it will be convenient to represent the v-variation of the surface current distribution by a series as follows:

$$K_\phi(v) = K(0)\ f(h_u, h_v, h_\phi)\ g(v) \sum_n \alpha_n V_n . \tag{22}$$

The functions f and g are inserted at this point for convenience in a later use of orthogonality properties. For the present, these functions are arbitrary (real) except that they may have no zeros in the range. K(0) is the value of the surface current density at the middle of the range of v (i.e., at the equator, z = 0). The functions in Eq. 22 must be such that the surface current density is finite everywhere and the total input current (Eq. 36) is finite. The coefficients and the functions V_n are real.

The boundary conditions may now be employed to determine the coefficients. From Eqs. 16, 18, and 19 we have

$$\sum_n a_n U_{n1}(u_o) V_n = \sum_n b_n U_{n2}(u_o) V_n$$

from which

$$a_n = \frac{U_{n2}(u_o)}{U_{n1}(u_o)} b_n . \tag{23}$$

From Eqs. 17, 20, 21 and 22 we have

$$\frac{1}{j\omega\mu_2 h_\phi h_u} \sum_n b_n \frac{dU_{n2}(u_o)}{du} V_n - \frac{1}{j\omega\mu_1 h_\phi h_u} \sum_n a_n \frac{dU_{n1}(u_o)}{du} V_n$$

$$= K(0) f(h_u, h_v, h_\phi) g(v) \sum_n \alpha_n V_n . \tag{24}$$

Thus, putting $dU_n/du = U_n'$ we find, from Eqs. 23 and 24, that

$$b_n = \frac{j\omega\mu_2 h_\phi h_u K(0) f(h_u, h_v, h_\phi) g(v) \alpha_n}{U_{n2}'(u_o)\left[1 - \dfrac{\mu_2}{\mu_1} \dfrac{U_{n2}(u_o)}{U_{n2}'(u_o)} \dfrac{U_{n1}'(u_o)}{U_{n1}(u_o)}\right]} . \tag{25}$$

The value of b_n inserted in Eq. 19 (or a_n in Eq. 18) gives E_ϕ, which is required for the impedance calculation.

5. A GENERAL EXPRESSION FOR THE IMPEDANCE

As was pointed out in the introduction, the impedance may be found from

$$|I_i|^2\, Z = -\iint_{\text{surface } u = u_0} E_\phi(u_0)\, K_\phi^*\, dS\,. \tag{26}$$

Writing $dS = h_v\, h_\phi\, dv\, d\phi$, and employing Eqs. 19 and 22, we find that

$$|I_i|^2 Z = -\iint \frac{1}{h_\phi} \sum_n b_n U_{n2}(u_0)\, V_n K^*(0)\, f(h_u, h_v, h_\phi)\, g(v) \sum_m \alpha_m V_m h_v h_\phi dv d\phi\,. \tag{27}$$

At this point we specify the functions f and g, which were inserted into the series expansion for the current. Note that if b_n from Eq. 25 is inserted explicitly into Eq. 27, the metrical coefficients can be grouped into a factor $(h_u\; h_v\; h_\phi)$. Thus, since $f(h_u, h_v, h_\phi)$ appears squared in that expression, a choice

$$f(h_u, h_v, h_\phi) = [h_u\; h_v\; h_\phi]^{-1/2} \tag{28}$$

frees the integral of metrical coefficients. Next, we choose the function g so that in the v integration in Eq. 27 we will have

$$\int_{\text{range}} dv\, [g(v)]^2\, V_n V_m = 0 \tag{29}$$

and

$$\int_{\text{range}} dv\, [g(v)]^2\, V_n^{\,2} = \Lambda_{V_n} \tag{30}$$

where Λ_{V_n} is a symbol for the normalization factor.

Provided that the variables separate in the manner indicated in Eqs. 8, 9 and 10 (less general form), the function g can now be specified. For suppose that two functions, V_k and V_n, represent two solutions to

Eq. 12; then we have

$$g_2(v)\, V_k'' + \{k + \omega^2\mu\varepsilon\, f_2(v)\}\, V_k = 0$$

and

$$g_2(v)\, V_n'' + \{n + \omega^2\mu\varepsilon\, f_2(v)\}\, V_n = 0 .$$

If the first equation is multiplied by V_n and the second by V_k and subtracted from the first,[2] we obtain

$$g_2(v) \frac{d}{dv} \left(V_n \frac{dV_k}{dv} - V_k \frac{dV_n}{dv} \right) + (k - n)\, V_k V_n = 0.$$

Dividing by $g_2(v)$ and integrating, we then have

$$\left(k - n\right) \int_{-v_o}^{v_o} \frac{V_k\, V_n}{g_2(v)}\, dv = V_k \frac{dV_n}{dv} - V_n \frac{dV_k}{dv} \Bigg|_{-v_o}^{v_o} = 0 ,$$

where the limits of integration are from pole to pole, so that if $k \neq n$, we have

$$\int_{-v_o}^{v_o} \frac{V_k\, V_n}{g_2(v)}\, dv = 0 , \qquad (31)$$

where $g_2(v)$ is the function which was specified in Eq. 8. Thus, if we choose

$$g(v) = g_2(v)^{-1/2} = \left[\frac{h_v}{h_\phi\, h_u} \right]^{+1/2} \qquad (32)$$

then Eq. 29 is satisfied since it is identical to Eq. 31.

When the values of f and g determined by Eqs. 28 and 32 are inserted into Eq. 27, the final result is

$$|I_i|^2 Z = -2\pi\, j\omega\mu_2\, [K(0)]^2 \int_{-v_o}^{v_o} \frac{dv}{g_2(v)} \frac{\sum\limits_n \alpha_n V_n \sum\limits_m \alpha_m V_m}{\left[\frac{U_{n2}'(u_o)}{U_{n2}(u_o)} - \frac{\mu_2}{\mu_1} \frac{U_{n1}'(u_o)}{U_{n1}(u_o)} \right]} . \qquad (33)$$

2. Here we follow Schelkunoff, *op. cit.*, p. 115.

To save writing, we will define a function

$$f_n(R,\mu) = \left[\frac{U_{n2}'(u_0)}{U_{n2}(u_0)} - \frac{\mu_2}{\mu_1}\frac{U_{n1}'(u_0)}{U_{n1}(u_0)}\right].$$

Note that the summation product indicated in Eq. 33 will consist of terms of two sorts, viz:

$$\frac{\alpha_n \alpha_m V_n V_m}{f_n(R,\mu_1)} \tag{a}$$

and

$$\frac{\alpha_n^2 V_n^2}{f_n(R,\mu_1)} . \tag{b}$$

By the orthogonality relation (Eq. 31), terms of the sort of (a) above will give zero when integrated over the range of v. Utilizing the definition (Eq. 30) for Λ_{V_n}, after interchanging the order of integration and summation, Eq. 33 becomes

$$|I_i|^2 Z = -2\pi j\omega\mu_2 \,[K(0)]^2 \sum_n \frac{\alpha_n^2 \Lambda_{V_n}}{f_n(R,\mu_1)} . \tag{35}$$

Next we relate the total input current to the surface current density at the equator, K(0). The input current must equal the current which crosses a half plane through the z-axis of the ferrite. The incremental contribution to this current is K(v) ds, where ds is an element of width along a v-curve (that is, along the locus of the intersection of the plane ϕ = constant, with the surface u = constant = u_0). Thus, since $ds = h_v dv$

$$I_i = \int K(v)\, h_v\, dv$$

or, from Eqs. 22, 28 and 32, after interchanging the order of summation and integration, we have

$$I_i = K(0) \sum_n \alpha_n \int_{\text{range}} dv \frac{h_v}{h_u h_\phi} V_n . \tag{36}$$

The integral in Eq. 36 is a definite integral (easily evaluated for the case of a sphere, at least). If we define

$$I(n) \equiv \int_{\text{range}} dv \frac{h_v}{h_u h_\phi} V_n , \tag{37}$$

then we have, from Eq. 36,

$$I_i = K(0) \sum_n \alpha_n I(n) . \tag{38}$$

Inserting Eq. 38 into Eq. 35 we have a general expression for the impedance

$$Z = \frac{-2\pi j\omega\mu_2}{\left[\sum_n \alpha_n I(n)\right]^2} \sum_n \frac{\alpha_n^2 \Lambda_{V_n}}{f(R,\mu_1)} \tag{39}$$

in which, by way of summary, the symbols are

$$\alpha_n = \frac{1}{K(0) \Lambda_{V_n}} \int_{\text{range}} dv\, K_v h_v V_n ,$$

the coefficients in the series expansion of the current,

$$\Lambda_{V_n} = \int_{\text{range}} dv \frac{h_v}{h_u h_\phi} V_n^2 ,$$

$$I(n) = \int_{\text{range}} dv \frac{h_v}{h_u h_\phi} V_n ,$$

$$f(R,\mu_1) = \left[\frac{U_{n2}'(u_0)}{U_{n2}(u_0)} - \frac{\mu_2}{\mu_1} \frac{U_{n1}'(u_0)}{U_{n1}(u_0)}\right].$$

The following form is usually more convenient:

$$Z = \frac{-2\pi \; j\omega\mu_1}{\left[\sum_n \alpha_n \; I(n)\right]^2} \sum_n \frac{U_{n1}\;(u_o)}{U_{n1}'(u_o)} \frac{\alpha_n^{\;2} \; \Lambda V_n}{\left[\frac{\mu_1}{\mu_2} \frac{U_{n2}'(u_o)}{U_{n2}\;(u_o)} \frac{U_{n1}\;(u_o)}{U_{n1}'(u_o)} - 1\right]} . \tag{40}$$

This expression is applicable to spheroids, or, in fact, to any shapes which meet the symmetry requirements and the conditions on the metrical coefficients.

6. APPLICATION TO A FERRITE SPHERE

We now return to the main problem of this report — namely, to find the input impedance of a ferrite sphere with latitudinal currents. The general expression (Eq. 40) can be applied directly. The angular and radial functions are known to be Legendre polynomials and spherical Bessel functions, respectively, (see Appendix A for details). The computation of the other quantities in the expression is simplified if a system of coordinates $(r, -\cos\theta, \phi)$ is selected, rather than the (r, θ, ϕ) system of Fig. 1. Thus, putting $v = -\cos\theta$, we find that the coordinates (r, v, ϕ) are related to the rectangular system by the equations

$$x = r(1 - v^2)^{1/2} \cos\phi,$$

$$y = r(1 - v^2)^{1/2} \sin\phi,$$

$$z = -rv$$

so that the metrical coefficients are

$$h_r = 1$$

$$h_v = r/(1 - v^2)^{1/2}$$

$$h_\phi = r\sqrt{1 - v^2} \tag{41}$$

as can be found from

$$h_i^{\,2} = \left(\frac{\partial x}{\partial \mu_i}\right)^2 + \left(\frac{\partial y}{\partial \mu_i}\right)^2 + \left(\frac{\partial z}{\partial \mu_i}\right)^2 \tag{42}$$

where μ_1 is successively r, v, and ϕ). Thus, since

$$V_n = (1 - v^2)^{1/2} P_n^{\,1} (v) \tag{43}$$

for the sphere ($P_n^{\,1}$ is the usual symbol for the associated Legendre polynomial) we find

$$\Lambda_{V_n} = \int dv \frac{h_v}{h_u \; h_\phi} V_n^{\,2} = \int dv \frac{1}{(1 - v^2)} [(1 - v^2)^{1/2} P_n^{\,1} (v)]^2$$

so that we can use a known result to find

$$\Lambda_{V_n} = \int_{-1}^{+1} dv \; [P_n^{\,1} (v)]^2 = \frac{2n(n + 1)}{2n + 1} . \tag{44}$$

Also

$$I (n) = \int dv \frac{h_v}{h_u \; h_\phi} V_n = \int_{1}^{+1} dv \; (1 - v^2)^{1/2} P_n^{\,1} (v)$$

[illegible] since

$$P_n^{\,1} (v) = (1 - v^2)^{1/2} \frac{dP_n(v)}{dv} ,$$

we have

$$I (n) = \int_{-1}^{+1} dP_n (v) = 2 . \tag{45}$$

The appropriate radial function for the inside of the ferrite (r < a) is

$$\hat{J}_n (\beta_1 r) = \sqrt{\pi\beta_1 r/2} \; J_{n + 1/2} (\beta_1 r) \tag{46}$$

and for the outside (r > a) is

$$\hat{K}_n (j\beta_2 r) = j^{\,(n+1)} \sqrt{\pi\beta r/2} \; [J_{n+1/2} (\beta r) - j \, N_{n+1/2} (\beta r)]. \tag{47}$$

Thus for a sphere of ferrite of radius a the impedance (from Eq. 40) is

$$Z = \frac{-2\pi j\omega\mu_1}{4(\sum_n \alpha_n)^2} \sum_n \frac{\hat{J}_n(\beta_1 a)}{\hat{J}_n{}'(\beta_1 a)} \frac{\alpha_n{}^2 \frac{2n(n+1)}{2n+1}}{\left[\frac{\mu_1}{\mu_2} \frac{\hat{K}_n{}'(j\beta_2 a)}{\hat{K}_n(j\beta_2 a)} \cdot \frac{\hat{J}_n(\beta_1 a)}{\hat{J}_n{}'(\beta_1 a)} - 1\right]} \tag{48}$$

in which everything is specified except the coefficients of the current expansion, α.

An explicit expression for the coefficients is obtained as follows: Let K(v) be a specified current distribution (for example, K(v) = K(0), $-\Delta \leq v \leq +\Delta$; K(v) = 0, otherwise). Then, by Eqs. 22, 28 and 32,

$$K(v) = K(0)\,[h_u\, h_\phi]^{-1} \sum_n \alpha_n V_n$$

or from Eqs. 41 and 43

$$K(v) = \frac{K(0)}{r(1 - v^2)^{1/2}} \sum_n \alpha_n (1 - v^2)^{1/2} P_n{}^1(v)$$

or

$$K(v) = \frac{K(0)}{a} \sum_n \alpha_n P_n{}^1(v). \tag{49}$$

Thus, a multiplication of Eq. 49 by $P_m{}^1$, and a subsequent use of the orthogonality properties of the Legendre polynomials shows that the coefficients are determined by

$$\alpha_n = \frac{a}{K(0)} \frac{2n+1}{2n(n+1)} \int_{-1}^{+1} dv\, K(v)\, P_n{}^1(v) \quad . \tag{49a}$$

Thus, Eq. 48 gives the input impedance in terms of completely specified quantities.

7. EXAMPLES OF PARTICULAR CURRENT DISTRIBUTIONS

The expression (48) for the input impedance will now be evaluated for two special cases.

7.1 Sinusoidal Current Distribution

One of the simplest (computationwise) current distributions is one in which

$$K(v) = K(0)\,(1 - v^2)^{1/2} = K(0) \sin\theta .$$

Such a distribution is of interest since it is known[3] that a sinusoidal distribution of direct current on a spherical surface gives a uniform magnetic field inside the sphere. Hence, the results of the present problem can be compared with the results of a familiar magnetostatic problem. Appropriate coefficients for such a distribution are $\alpha_n = a$, $n = 1$ and $\alpha_n = 0$, $n \neq 1$. Thus, the series expression (48) for the impedance reduces to a single term as follows:

$$Z = -\,2/3\ \pi j \omega \mu_1 \frac{\hat{J}_1\,(\beta_1 a)}{\hat{J}_1{}'(\beta_1 a)} \left[\frac{\mu_1}{\mu_2}\, \frac{\hat{K}_1{}'(j\beta_2 a)}{\hat{K}_1\,(j\beta_2 a)} \cdot \frac{\hat{J}_1\,(\beta_1 a)}{\hat{J}_1{}'(\beta_1 a)} - 1 \right]^{-1} . \qquad (50)$$

This expression can be evaluated exactly by inserting

$$\hat{J}_1(\beta a) = \frac{\sin\beta_1 a}{\beta_1} - \cos\beta_1 a, \quad \hat{J}_1{}'(\beta a) = -\frac{1}{a}\left(\frac{\sin\beta_1 a}{\beta_1 a} - \cos\beta_1 a\right) + \sin\beta_1 a$$

$$\hat{K}_1(j\beta_2 a) = e^{-j\beta_2 a}\left(1 + \frac{1}{j\beta_2 a}\right), \quad \hat{K}_1{}'(j\beta_2 a) = -j\beta_2 \hat{K}_1(j\beta_2 a) + \frac{je^{j\beta_2 a}}{\beta_2 a^2}$$

so that

$$\frac{\hat{J}_1{}'(\beta_1 a)}{\hat{J}_1\,(\beta_1 a)} = -\frac{1}{a} + \frac{\beta_1{}^2 a}{1 - \beta_1 a \cot\beta_1 a} \qquad (51)$$

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3 Smythe, W. R. *Static and Dynamic Electricity*, McGraw-Hill Book Co., Inc., New York, N. Y. 2nd Edition, 1950 p. 274.

and

$$\frac{\hat{K}_1'(j\beta_2 a)}{\hat{K}_1\ (j\beta_2 a)} = -j\beta_2 + \frac{j}{a}\frac{1}{(\beta_2 a - j)} = \frac{-(1 + j\beta_2{}^3 a^3)}{a[1 + (\beta_2 a)^2]} . \tag{52}$$

The problem of interest at the present time is that of the small ferrite antenna, that is $\beta a << 1$. Writing the series expansion for $\hat{J}_1(\beta a)$(or Eq. 51) and retaining only the terms in the lowest order of (βa) gives

$$\frac{\hat{J}_1\ (\beta_1 a)}{\hat{J}_1'(\beta_1 a)} \simeq \frac{a}{2} . \tag{53}$$

Similarly, from Eq. 52,

$$\frac{\hat{K}_1'(j\beta_2 a)}{\hat{K}_1\ (j\beta_2 a)} \simeq -\frac{1}{a} , \tag{54}$$

for a non-radiating approximation.

Typically, the medium of Region 2 (outside the ferrite) will be air, so that the ratio μ_1/μ_2 is $\mu_1/\mu_2 = \mu_1/\mu_0 = K_m$, the relative permeability.

Substitution of these values in Eq. 50 gives, for a non-radiating approximation,

$$Z \simeq \frac{\pi}{3}\, j\omega\mu_0 a \frac{K_m}{\left(\frac{K_m}{2} + 1\right)} . \tag{55}$$

The ratio of the impedance of a ferrite sphere to that of a sphere having $K_m = 1$ is seen to be

$$\frac{Z_f}{Z_o} = \frac{3K_m}{K_m + 2} . \tag{56}$$

Note that the limit of this ratio as $K_m \to \infty$, is equal to 3, and, at $K_m = 100$, it has the value 2.94.

As was pointed out in Section 4 the permeability of the ferrite may be complex, $K_m = K_m' - jK_m''$. In this case, Z has both a real and

imaginary component as follows:

$$Z \simeq \frac{2}{3}\pi j\omega\mu_o a \; \frac{(K_m' - jK_m'')}{K_m' + 2 - jK_m''} \quad .$$

$$\approx \frac{2}{3}\pi j\omega\mu_o a \left\{ \frac{K_m'}{K_m' + 2} - \frac{2jK_m''}{(K_m + 2)^2} \right\} \tag{57}$$

if $K_m'^2 >> K_m''^2$.

If the resistance of the conductor is negligible (as has been assumed in the derivation of the impedance), a quality factor may be found as follows:

$$Q = \frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)} = \frac{K_m'(K_m' + 2)}{2K_m''} \quad . \tag{58}$$

The radiation resistance may be calculated by carrying an additional term in the approximation for the exterior radial function. We will, therefore, approximate Eq. 52 by

$$\frac{\hat{K}_1'(j\beta_2 a)}{\hat{K}_1\ (j\beta_2 a)} \doteq - \frac{(1 + j(\beta_2 a)^3}{a} \tag{59}$$

and Eq. 51 by

$$\frac{\hat{J}_1\ (\beta_1 a)}{\hat{J}_1'(\beta_1 a)} \doteq ½a(1 - 3/5(\beta_1 a)^2) \doteq ½a \ . \tag{60}$$

The latter approximation is not consistent, but the factor $(1-3/5(\beta_1 a)^2)$ will not change the numerical results appreciably so it is omitted to save algebra. The insertion of Eqs. 59 and 60 into Eq. 50 results in

$$Z \doteq \frac{\pi}{3} j\omega\mu_o a \left[\frac{K_m}{K_m \dfrac{(1 + j(\beta_2 a)^3)}{2} + 1} \right]. \tag{61}$$

If K_m is real, a rearrangement gives

$$Z \doteq \frac{\pi}{3} j\omega\mu_o a \left[\frac{K_m(K_m + 1) - j \frac{K_m^2}{2} (\beta_2 a)^3}{(\frac{K_m}{2} + 1)^2 + K_m^2 \frac{(\beta_2 a)^6}{4}} \right] \tag{62}$$

$$\doteq \frac{\pi}{3} j\omega\mu_o a \left[\frac{K_m}{(\frac{K_m}{2} + 1)} - \frac{j \frac{K_m^2}{2} (\beta_2 a)^3}{(\frac{K_m}{2} + 1)^2} \right]. \tag{63}$$

The real part of this expression is the radiation resistance

$$R_R \doteq \frac{\pi}{6} \omega\mu_o a [K_m / (\frac{K_m}{2} + 1)]^2 (\beta_2 a)^3 . \tag{64}$$

Note that the ratio of radiation resistance of a ferrite sphere to that of a sphere having the permeability of free space is the same as the square of the ratio (Eq. 56). The limit of the ratio as $K_m \to \infty$ is 9. When $K_m = 100$, the value of the ratio is 8.6.

If K_m is complex, we find, from Eq. 61,

$$Z \doteq \frac{\pi}{3} j\omega\mu_o a \left[\frac{K_m'}{(\frac{K_m'}{2} + 1)} - \frac{jK_m''}{(\frac{K_m'}{2} + 1)^2} - \frac{j \frac{K_m'^2}{2} (\beta_2 a)^3}{(\frac{K_m'}{2} + 1)^2} \right], \tag{65}$$

wherein additional approximations have been made as follows:

$$\left(\frac{K_m''}{2}\right)^2 \{1 - \frac{K_m'}{K_m''} (\beta_2 a)^3\} << (\frac{K_m}{2} + 1)^2$$

and

$$\left(\frac{K_m''}{2}\right)^2 \{1 - \frac{K_m'}{K_m''} (\beta_2 a)^3\} << K_m' (\frac{K_m'}{2} + 1) .$$

Equation 65 shows two contributions to the real part of the impedance. These have their origins in power dissipated in the ferrite and power radiated, respectively. Thus, a ferrite antenna radiation efficiency

may be found as follows:

$$\text{Efficiency} = \frac{R_R}{R_c + R_R} = \frac{1}{1 + (R_c/R_R)} \doteq \frac{1}{1 + \frac{2K_m''}{(K_m')^2 \, (\beta_2 a)^3}} \; ,$$

or

$$\text{Efficiency} \doteq \frac{(K_m')^2 \, (\beta_2 a)^3}{2(K_m'')} = \frac{(K_m')^2}{K_m''} \, 3\pi^2 \, \frac{V}{\lambda^3} \qquad (66)$$

where V is the volume of the sphere, and R_c is effective ferrite resistance.

As was pointed out earlier, the assumption of a current distribution $K(v) = K(0) \sin\theta$ allows the intermediate steps of the generalized impedance calculation to be checked by a comparison with known results of a magnetostatic problem. Employing Eqs. 13, 14, 23, 25, 28, 32, 53 and 54, one finds for the magnetic field inside the ferrite

$$H_r = 2K(0) \cos\theta \, \frac{\mu_2}{(\mu_1 + 2\mu_2)} \; ,$$

$$H_\theta = -2K(0) \sin\theta \, \frac{\mu_2}{(\mu_1 + 2\mu_2)} \; .$$

Thus we find

$$H_z = H_0 \, \frac{3\mu_2}{\mu_1 + 2\mu_2} \; , \quad \text{where } H_0 = \frac{2K(0)}{3} \; ,$$

which checks with the corresponding static solution.

In view of this agreement, it is tempting to express the results in terms of the known static demagnetization factor for a sphere, $D_s = 1/3$. If this is done, the results are in a convenient form for extrapolated application to other shapes for which the static demagnetization factor is known (namely, ellipsoids and rods). Then, the ratio (Eq. 56)

becomes

$$\frac{Z_f}{Z_o} = \frac{K_m}{1 + \frac{1}{3}(K_m - 1)} = \frac{K_m}{1 + D_s(K_m - 1)} \tag{67}$$

and the limit of this ratio as $K_m \to \infty$ is $1/D_s$. In a similar fashion, the quality factor is

$$Q = \frac{K_m'(1 + D_s(K_m - 1))}{K_m''(1 - D_s) + \left(\frac{K_m'}{3}\right)^2 (\beta_2 a)^3} \tag{68}$$

while the efficiency is

$$\mathcal{E} = \frac{K_m'^2}{K_m''} \frac{2\pi^2}{1 - D_s} \frac{V}{\lambda^3} . \tag{69}$$

In general, demagnetization factors are smaller for elongated shapes than for flat stubby shapes.

7.2 An Approximation to a Current Band

The effect of a concentration of current (or windings) in the equatorial region is studied by determining an appropriate set of current coefficients. A convenient set is $\alpha_1 = a$, $\alpha_3 = -a/3$, $\alpha_5 = .086a$, and $\alpha_n = 0$, $n \neq 1$, 3, or 5. The current distribution which is determined by these coefficients is plotted in Fig. 2. Once again, let us be content with approximations for the radial functions which are valid for small spheres ($\beta a < .2$). For this case,

$$\hat{J}_n \doteq \frac{2^n\, n!}{(2n + 1)!} (\beta_1 r)^{n+1} \tag{70}$$

and in the non-radiating approximation

$$\hat{K}_n \doteq -j \frac{2(n)!}{2^n\, n!} (\beta_2 r)^{-n} \tag{71}$$

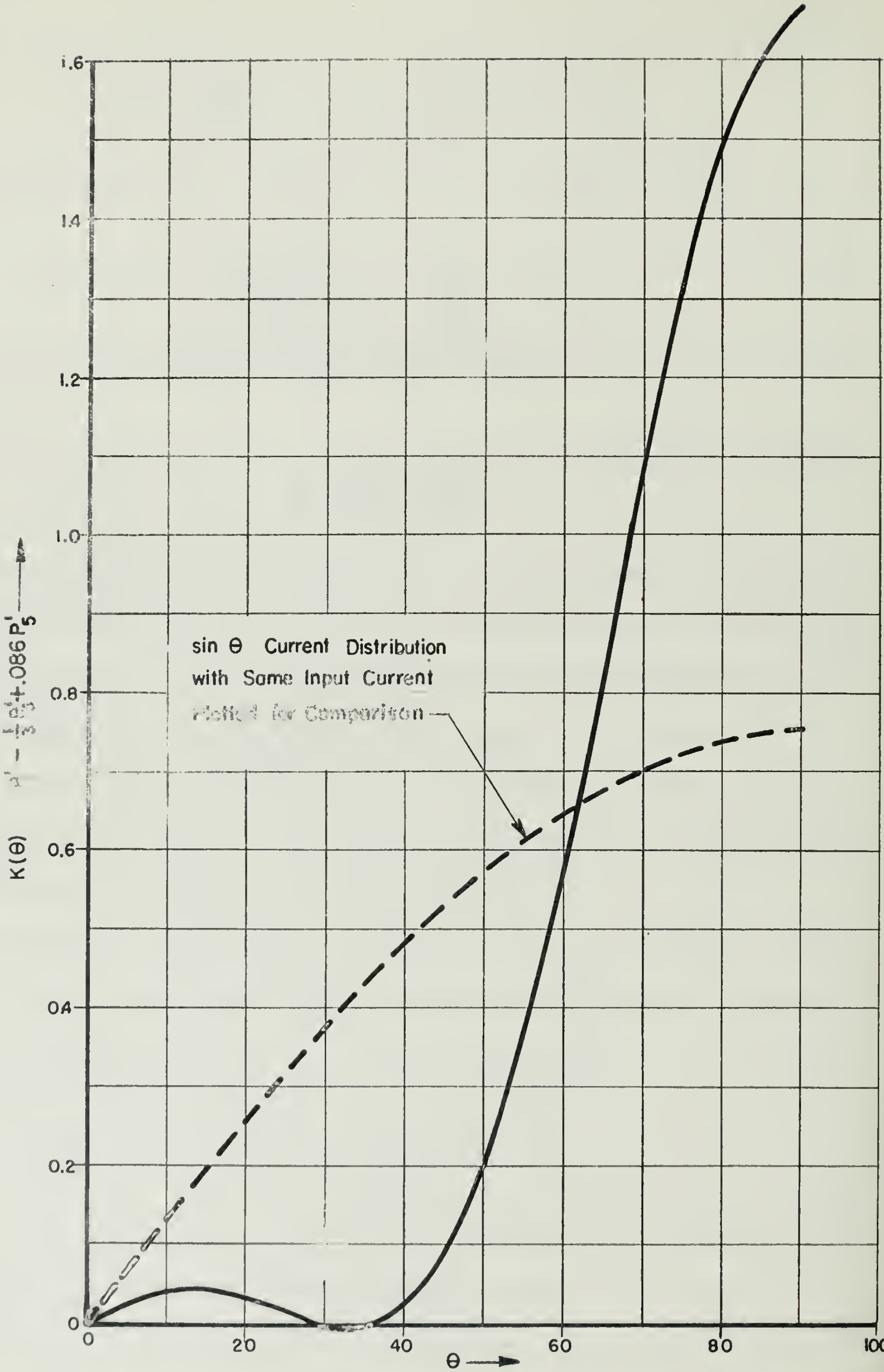

Figure 2 An Approximation to a Current Band

so that

$$\frac{\hat{J}_n(\beta_1 a)}{\hat{J}_n{}'(\beta_1 a)} \doteq \frac{a}{n+1} \tag{72}$$

and

$$\frac{\hat{K}_n{}'(j\beta_2 a)}{\hat{K}_n(j\beta_2 a)} \doteq \frac{n}{a} . \tag{73}$$

Thus, Eq. 48 becomes

$$Z = \frac{\pi}{2}\,\frac{j\omega\mu_1}{(.56)}\,a \sum_n \alpha_n{}^2\,\frac{2n}{2n+1}\,\frac{1}{\left[\dfrac{\mu_1}{\mu_2}\,\dfrac{n}{n+1} + 1\right]}$$

$$= \frac{\pi j\omega\mu_0 a}{(.56)}\left[\frac{1}{3}\,\frac{K_m}{\dfrac{K_m}{2}+1} + \frac{1}{9}\,\frac{3}{7}\,\frac{K_m}{(\frac{3}{4}K_m + 1)} + 7.4\times 10^{-3}\,\frac{5}{11}\,\frac{K_m}{(5/6K_m + 1)}\right]. \tag{74}$$

The effect of the current concentration can be determined by comparing Eqs. 74 and 55. (It is apparent that the absolute value of the impedance is greater with an equatorial current concentration whatever the value of the permeability, but this is not the point of main interest here.) The ratio of impedances with and without ferrite material (as in Eq. 56) is

$$\lim_{K_n\to\infty}\frac{Z_f}{Z_o} \doteq 2.92 ,$$

as compared to the value of three with a sinusoidal current distribution. The values of this ratio for other values of permeability are given in Table 1.

Impedance Ratios Z_f/Z_o (K_m real)

Permeability	∞	100		10		2	
		Z_f/Z_o	% max	Z_f/Z_o	% max	Z_f/Z_o	% max
Uniform Field (sin θ Distribution)	3.00	2.94	98	2.5	83	1.5	50
Approximation to Band	2.92	2.86	98-	2.47	84	1.51	52

Table I

The effect on the quality factor can be assessed by allowing the permeability in Eq. 74 to become complex. Table 2 gives the ratio $Q = \text{Im } Z/\text{Re } Z$ for different permeability values. To make the results comparable, the loss tangent was assumed to be the same in each case with the value $\tan \delta = K_m''/K_m' = .01$.

Quality Factor for a Ferrite Sphere

Permeability	100		10		2	
	Q	$\frac{Q_{sphere}}{Q_{toroid}}$	Q	$\frac{Q_{sphere}}{Q_{toroid}}$	Q	$\frac{Q_{sphere}}{Q_{toroid}}$
Uniform Field ($\sin\theta$ Distribution)	5100	51	600	6.	200	2
Approximation to Band	5240	52.4	614	6.14	207	2.07

Table 2

The ratio of the Q for a ferrite sphere to the corresponding Q for a ferrite toroid with a current sheet winding is included for comparison. (The inverse is, of course, the ratio of the effective loss tangents.) Note that the current concentration improves the Q slightly (copper loss neglected).

The effect of the concentration of current on the radiation resistance is studied by replacing the radial function (Eq. 73) by a slightly better approximation as follows:

$$\frac{\hat{K}_1'(j\beta_2 a)}{\hat{K}_1(j\beta_2 a)} = -\frac{1}{a}\frac{(1 + j(\beta_2 a)^3)}{1 + (\beta_2 a)^2} \doteq -\frac{1}{a}[1 + j(\beta_2 a)^2] \tag{75}$$

$$\frac{\hat{K}_3'(j\beta_2 a)}{\hat{K}_3(j\beta_2 a)} \doteq -\frac{1}{a}\frac{(3 + 2(\beta_1 a)^2 + j(\beta_2 a)^3)}{1 + (\beta_2 a)^2} \doteq -\frac{1}{a}[3 + j(\beta_2 a)^3] \tag{76}$$

$$\frac{\hat{K}_5'(j\beta_2 a)}{\hat{K}_5(j\beta_2 a)} \doteq -\frac{1}{a}\frac{(5 + 4(\beta_2 a)^2 + j(\beta_2 a)^3)}{1 + (\beta_2 a)^2} \doteq -\frac{1}{a}[5 + j(\beta_2 a)^3] \; . \tag{77}$$

Employing Eqs. 75, 76, and 77 the expression for the impedance becomes

$$Z \doteq \frac{\pi j \omega \mu_0 a}{[.56]} \left[\frac{1}{3} \frac{K_m}{\left(\frac{K_m}{2} + 1\right) + j \frac{K_m}{2} (\beta_2 a)^3} + \frac{3}{63} \frac{K_m}{(3/4 K_m + 1) + j \frac{K_m}{4} (\beta_2 a)^3} \right.$$

$$\left. + \frac{37}{11} \times 10^{-3} \frac{K_m}{(5/6 K_m + 1) + j \frac{K_m}{6} (\beta_2 a)^3} \right].$$

Table 3 shows the very small effect on the radiation resistance. The table exhibits the ratio of a ferrite filled spherical current, R_{Rf} to the same current established on a sphere with permeability 1, R_{Ro}.

Permeability	∞	100	10	2	
Uniform Field (sin θ Distribution)	9	8.65	6.25	2.25	R_{Rf}/R_{Ro}
Approximation to Band	8.92	8.45	6.14	2.24	R_{Rf}/R_{Ro}

Table 3

8. THE NEARLY SPHERICAL SPHEROID

Equation 40 is valid for spheroids as well as for spheres; only the functions are different. Furthermore, as is pointed out in Appendix B, if the spheroids are nearly spherical, the spheroidal functions approach those which correctly describe the sphere. Thus it is possible to predict the results of calculations for nearly spherical spheroids. Since the fields for small spheres check with those calculated for corresponding static fields, we can expect that the static field solutions for spheroids in a uniform field will be equally valid. These solutions are known and tabulated in terms of demagnetization factors.[4] The demagnetization factors for prolate spheroids, spheres and oblate spheroids increase in that order. The impedance ratios expressed in terms of the demagnetization factor are given in Eqs. 67, 68, and 69. Note that, according to Eqs. 68 and 69, the Q and, to a much smaller extent, the radiation efficiency increase with increasing demagnetization factor (copper loss negligible).

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4 Osborn, J. A. *Phys. Rev. 67*, p. 351, 1945.

9. SUMMARY AND CONCLUSIONS

A formula for the input impedance of a permeable spheroid with a latitudinal surface current has been derived and evaluated for the case of a small sphere having two simple surface current distributions. The variation of the magnitude of the surface current with angle (θ) was represented by a series of associated Legendre polynomials. The effect of concentrating the current in the equatorial region was considered; there appears to be a small advantage in such a current concentration (neglecting copper losses). Formulas for quality factor and radiation efficiency are obtained. The results obtained for the problem of the small sphere are extrapolated to the case of a small spheroid, and the advantage of flattened shapes is pointed out.

APPENDIX A

THE FORM OF THE DERIVATION FOR THE PARTICULAR CASE OF THE SPHERE

The general expression for the impedance which was derived in Sections 2 to 5 of this report is applicable to antennas with shapes which coincide with some surface of orthogonal coordinates with symmetry about the z-axis. The parallel argument for the particular case of a sphere is presented here for the purpose of illustration.

In the spherical system of coordinates, the metrical coefficients are $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$. Hence, the field equations (1, 2 and 3) become

$$\frac{\partial(r \sin\theta\, E_\phi)}{\partial\theta} = j\omega\mu r^2 \sin\theta\, H_r \tag{A-1}$$

$$\frac{\partial(r \sin\theta\, E_\phi)}{\partial\theta} = j\omega\mu r \sin\theta\, H_\theta \tag{A-2}$$

$$\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta} = j\omega\varepsilon r E_\phi. \tag{A-3}$$

To help in the solution of this system of equations, introduce $A = r \sin\theta\, E_\phi$ Then

$$H_r = -\frac{1}{j\omega\mu r^2 \sin\theta}\,\frac{\partial A}{\partial\theta} \tag{A-4}$$

$$H_\theta = \frac{1}{j\omega\mu r \sin\theta}\,\frac{\partial A}{\partial r} \tag{A-5}$$

$$E_\phi = \frac{A}{r \sin\theta}. \tag{A-6}$$

The insertion of these equations into Eq. A-3 gives

$$\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial A}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial A}{\partial\theta}\right) + \frac{\omega^2\mu\varepsilon A}{\sin\theta} = 0$$

Put $A = R(r)\,\Theta\,(\theta)$ to separate variables. Then

$$\frac{1}{\sin\theta}\,\Theta\,\frac{\partial^2 R}{\partial^2 r} + \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\right) R\,\frac{\partial\Theta}{\partial\theta}$$

$$+ \frac{1}{r^2\sin\theta}\,R\,\frac{\partial^2\Theta}{\partial\theta^2} + \frac{\omega^2\mu\varepsilon R\Theta}{\sin\theta} = 0 \quad . \qquad \text{(A-7)}$$

Multiplication by $r^2 \sin\theta/R\Theta$ results in

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Theta}{\partial\theta}\right) + \omega^2\mu\varepsilon r^2 = 0.$$

(Note that in this case, the more general condition in Eq. 9 was satisfied so that $g_1(u) = r^2$ and $g_3(v) = \sin\theta$. If we had employed the coordinates $(v, -\cos\theta, \phi)$ as in Section 6, then the simpler condition (Eq. 9), $h_u/h_v\, h_\phi$, being a function of u only, would have obtained.) The variables are now separated and we have the two equations.

$$R'' + (\omega^2\mu\varepsilon - \frac{k}{r^2})\,R = 0 \qquad \text{(A-11)}$$

and

$$\sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Theta}{\partial\theta}\right) + k\,\Theta = 0 \quad . \qquad \text{(A-12)}$$

Equation A-11 is immediately recognized as a modified Bessel equation for which the solutions are the spherical Bessel functions such as those given in Eqs. 46 and 47. A further transformation of Eq. A-12 is

helpful. If we introduce $v = -\cos\theta$, Eq. A 12 becomes

$$(1 - v^2)\frac{\partial^2 \Theta}{\partial v^2} + k\,\Theta = 0 \quad . \qquad \text{(A-12a)}$$

Then the substitution $\Theta = (1 - v^2)^{\frac{1}{2}}\bar{V}$ gives

$$(1 - v^2)\frac{d^2\bar{V}}{dv^2} - 2v\frac{d\bar{V}}{dv} + \left(k - \frac{1}{1 - v^2}\right)\bar{V} = 0,$$

which is immediately recognized as the associated Legendre equation (with $m = 1$ and $k = n(n + 1)$). Hence $\bar{V}_k = P_n^{\,1}(v)$ and

$$\Theta \equiv V_n = (1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v) = \sin\theta\, P_n^{\,1}(-\cos\theta)$$

as was pointed out in Eq. 43 of this report.

Application of the boundary conditions (16 and 17) at the surface $r = a$ results in the equation

$$\sum_n a_n \hat{J}_n(\beta_1 a)(1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v) = \sum_n b_n \hat{K}_n(j\beta_2 a)(1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v)$$

so that

$$a_n = \frac{\hat{K}_n(j\beta_2 a)}{\hat{J}_n(\beta_1 a)}\, b_n \quad , \qquad \text{(A-23)}$$

and the equation

$$\frac{1}{j\omega\mu_2 r\sin\theta}\sum_n b_n \hat{K}_n'(j\beta_2 a)(1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v)$$

$$- \frac{1}{j\omega\mu_1 r\sin\theta}\sum_n a_n \hat{K}_n'(j\beta_2 a)(1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v)$$

$$= K(0)\, f(h_r \cdot h_v \cdot h_\phi)\, g(v)\sum_n \alpha_n (1 - v^2)^{\frac{1}{2}} P_n^{\,1}(v) \qquad \text{(A-24)}$$

so that

$$b_n = \frac{j\omega\mu_2 r \sin\theta\, K(0)\, f\,(h_r, h_v, h_\phi)\, g\,(v)\, \alpha_n}{\hat{K}_n{}'\,(j\beta_2 a)\left[1 - \frac{\mu_2}{\mu_1}\frac{\hat{K}_n(j\beta_2 a)}{\hat{K}_n\,(j\beta_2 a)}\frac{\hat{J}_n{}'\,(\beta_1 a)}{\hat{J}_n(\beta_1 a)}\right]} . \qquad \text{(A-25)}$$

The impedance expression, Eq. 27, is then

$$I_i^2\, Z = -\iint \frac{1}{r(1 - v^2)^{1/2}} \sum_n b_n \hat{K}_n\,(j\beta_2 a)\,(1 - v^2)^{1/2} \cdot$$

$$P_n{}^1(v)\, \overset{*}{K}(0)\, f\,(h_r, h_v, h_\phi)\, g\,(v) \sum_m \alpha_m V_m r^2\, dv\, d\phi \quad . \qquad \text{(A-27)}$$

Choose

$$f\,(h_r, h_v, h_\phi) = [r^2]^{-1/2} . \qquad \text{(A-28)}$$

Choose g(v) so that

$$\int_{-1}^{+1} dv\, [g\,(v)]^2\, (1 - v^2)\, P_n{}^1\, P_m{}^1 = 0 \quad . \qquad \text{(A-29)}$$

This suggests

$$g\,(v) = (1 - v^2)^{-1/2} = \left[\frac{h_v}{h_\phi\, h_r}\right]^{1/2} \quad . \qquad \text{(A-32)}$$

Then

$$\int_{-1}^{+1} dv\, [g\,(v)]^2\, (1 - v^2)\, (P_n{}^1)^2 = \Lambda_{V_n} = \frac{2n(n + 1)}{2n + 1} \quad . \qquad \text{(A-30)}$$

Equation 35 for the case of a sphere is

$$I_i^2\, Z = -2\pi j\omega\mu_2\, [K(0)]^2 \sum_n \frac{\alpha_n{}^2\, 2n(n + 1)}{(2n + 1)\left[\frac{\hat{K}_n{}'(j\beta_2 a)}{\hat{K}_n\,(j\beta_2 a)} - \frac{\mu_2}{\mu_1}\frac{\hat{J}_n{}'(\beta_1 a)}{\hat{J}_n\,(\beta_1 a)}\right]} .$$

(A-35)

The input current in the case of a sphere of values a is

$$I_i = \int_0^{\pi} K(\theta)\ a\ d\theta.$$

Since K is more conveniently expressed as a function of v rather than θ, the change of variables is incorporated here to give

$$I_i = \int_{-1}^{+1} K(v)\ \frac{a dv}{\sqrt{1 - v^2}}$$

$$= \int_{-1}^{1} K(0) \sum_n \alpha_n \frac{P_n^{\,1}(v)}{(1 - v^2)^{1/2}}\ dv \qquad \text{(A-36)}$$

or ,

$$I_i = K(0) \sum_n \alpha_n \int_{-1}^{+1} \frac{P_n^{\,1}(v)}{(1 - v^2)^{1/2}}\ dv$$

$$= K(0) \sum_n \alpha_n \int_{-1}^{+1} d\ P_n\ (v)$$

$$= K(0)\ 2 \sum_n \alpha_n\ . \qquad \text{(A-38)}$$

Substitution of Eq. A-38 into Eq. A-35 gives Eq. 48 of Section 6. The parallel argument is thus completed.

If the problem of interest is that of a sphere with a number of turns of fine wire in series (all carrying the same current) then

$$K_\phi \simeq nI_i$$

where n is the number of turns per unit length of arc (meridian) and I_i is the current in one turn. Thus we have

$$\int K\ ds = \int n\ I_i\ ds = I_i \int nds = I_i N$$

so that the quantity I_i in the main part of this report is equal to $\int K ds/N$. Therefore each term in the expression for the impedance is multiplied by N^2, where N is the total number of turns.

APPENDIX B

SOME DETAILS FOR THE PROBLEM OF THE FERRITE SPHEROID

The general expression for the impedance, Eq. 40, is valid for spheroidal coordinate systems.

B I Prolate Spheroid

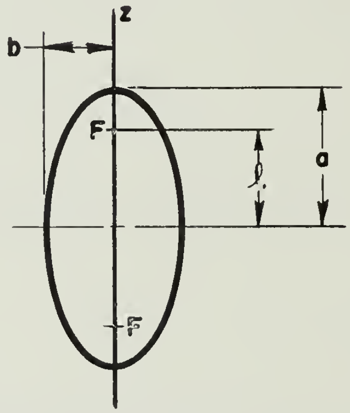

Figure 3

Let us consider the prolate spheroidal coordinate system in which the coordinate surfaces u = constant and v = constant represent, respectively, families of confocal prolate spheroids and hyperboloids of two sheets. The numbers u and v are, respectively, the reciprocals of the eccentricities of the generating ellipses and hyperbolas. The metrical coefficients are

$$h_u = l\sqrt{\frac{u^2 - v^2}{u^2 - 1}}, \quad h_v = l\sqrt{\frac{u^2 - v^2}{1 - v^2}}, \quad h_\phi = l\sqrt{(u^2 - 1)(1 - v^2)}$$

where $l = ae = a/u$ is the distance from the center to the focus of the ellipsoid, a is the semi-major axis, and e is the eccentricity. The semi-minor axis, b, is given by $b = l\sqrt{u^2 - 1}$.

The differential equations satisfied by U and V are found from Eqs. 8, 9, 10, 11, and 12:

$$g_2(v) = l(1 - v^2) \tag{B-8}$$

$$g_1(u) = l(u^2 - 1) \qquad \text{(B-9)}$$

so

$$f_1(u) = l^3 u^2 , \quad f_2(v) = -l^3 v^2 \qquad \text{(B-10)}$$

and

$$(u^2 - 1) U'' + (\omega^2 \mu\varepsilon\, l^2 u^2 - k) U = 0 \qquad \text{(B-11)}$$

$$(1 - v^2) V'' + (k - \omega^2 \mu\varepsilon\, l^2 v^2) V = 0 \, . \qquad \text{(B-12)}$$

It appears, therefore, that the "radial" functions U and the "angular" functions V satisfy the same differential equation. The substitution $V = (1 - v^2)^{1/2} \bar{V}$ transforms Eq. B-12 into a differential equation which becomes the associated Legendre equation as $\beta^2 l^2 \to 0$. The functions which satisfy these equations are known and partially tabulated.[5,6]

The other quantities in the impedance expression are:

$$\Lambda_{V_n} = \frac{1}{l} \int_{-1}^{1} dv \, \frac{V_n^{\,2}}{(1 - v^2)} = \frac{1}{l} \int_{-1}^{1} dv \, \bar{V}_n^{\,2}$$

where

$$V_n = (1 - v^2)^{1/2} \bar{V}_n \, ,$$

and

$$I(n) = \frac{1}{l} \int_{-1}^{1} dv \, \frac{V_n}{(1 - v^2)} \, .$$

The current distribution is represented by a series as follows (Eqs. 22, 28, and 32)

$$K_\phi(v) = \frac{K(0) \sum_n \alpha_n V_n}{l^2 \sqrt{(u^2 - v^2)}\,(1 - v^2)} = \frac{K(0) \sum_n \alpha_n \bar{V}_n}{l^2 (u^2 - v^2)^{1/2}}$$

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5. Stratton, Morse, Chu and Hutner *Elliptic Cylinder and Spherical Wave Functions*, The Technology Press, Massachusetts Institute of Technology, in conjunction with John Wiley & Sons, Inc., New York. 1941.

6. Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics*, McGraw-Hill Book Co., Inc., New York, New York. p. 1502 ff. and pp. 1576-1579.

so that the coefficients may be determined from

$$\alpha_n = \frac{l}{K(0)\ \Lambda_{V_n}} \int dv\ (u^2 - v^2)^{1/2} K_\phi(v)\ \bar{V}_n \ .$$

Thus, a formal procedure is prescribed for determining the input impedance of a ferrite spheroid with a surface which coincides with the prolate spheroidal coordinate surface u = constant.

B.2 Oblate Spheroidal Coordinates

Let us consider the oblate spheroidal coordinate system in which the coordinate surfaces u = constant and v = constant represent, respectively, families of confocal oblate spheroids and hyperboloids of one sheet. The numbers u and v are the reciprocals of the eccentricities of the generating ellipses and hyperbolas. The metrical coefficients are

$$h_u = l\left(\frac{u^2 + v^2}{u^2 + 1}\right)^{1/2}, \quad h_v = l\left(\frac{u^2 + v^2}{1 - v^2}\right)^{1/2}, \quad h_\phi = l\ [(u^2 + 1)\ (1 - v^2)]^{1/2}$$

where again l is the distance from the center of the spheroid to its focus. The differential equations for U and V are found from Eqs. 8, 9, 10, 11, and 12

$$(u^2 + 1)\ U'' + (\omega^2\mu\varepsilon\ l^2\ u^2 - k)\ U = 0$$

$$(1 - v^2)\ V'' + (k + \omega^2\mu\varepsilon\ l^2\ v^2)\ V = 0\ .$$

The transformation $u = j\bar{u}$ makes the equations for U and V identical. The substitution $V = (1 - v^2)^{1/2}\ \bar{V}$ transforms the "angular" equation into an equation for $\bar{V}$ which approaches the associate Legendre equation as $\beta^2 l^2 \to 0$. The rest of the analysis closely parallels that of the prolate spheroid. Only the functions are different.